$H \subseteq G$ normal $\iff$ $v \in \mathfrak{h},\ w \in \mathfrak{g} \Rightarrow [v,w] \in \mathfrak{h}$ ($\mathfrak{h}$ ideal)

Adjoint rep Lie group: $\quad Ad_g : G \to G$, $\quad h \mapsto ghg^{-1}$

$Ad_g := d(Ad_g)_e : \mathfrak{g} \to \mathfrak{g}$, $\quad v \mapsto \frac{d}{dt}\left(g\exp(tv)g^{-1}\right)\big|_{t=0}$

$\to Ad : G \longrightarrow GL(\mathfrak{g})$

Ad rep Lie algebra $\to$ $ad := d(Ad)_e : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$

$w \longrightarrow d(ad)_e(w) : \mathfrak{g} \to \mathfrak{g}$

$v \to d(ad)_e(w)(v) =$

$= \frac{d}{ds}\frac{d}{dt} \exp(sw)\exp(tv)\exp(-sw)$

$= [v,w]$

Si $H \subseteq G$ normal,

$ad : G \longrightarrow GL(\mathfrak{h})$

$g \longrightarrow ad_g : \mathfrak{h} \to \mathfrak{h}$

$v \to \frac{d}{ds}\left(g\exp(sv)g^{-1}\right)_{s=0}$

$d(ad)_e : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{h})$

$w \longrightarrow d(ad)_e(w) : \mathfrak{h} \to \mathfrak{h}$

$v \longrightarrow \frac{d}{ds}\frac{d}{dt} \cdots = [w,v] \in \mathfrak{h}$ (ideal)

Recíprocamente, supongamos $v \in \mathfrak{h} \Rightarrow [w,v] \in \mathfrak{h}$

Claim: Sea $w \in \mathfrak{g}$, $v \in \mathfrak{h}$, $\exp(w)\exp(v)\exp(-w) \in H$

Proof: $\exp : \mathfrak{gl}(\mathfrak{g}) \longrightarrow GL(\mathfrak{g})$

$f \longrightarrow \exp(f) = 1 + f + \frac{1}{2}f^2 + \cdots$

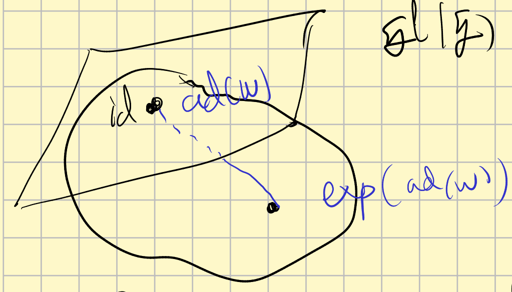
$ad(w) \in \mathfrak{gl}(\mathfrak{g})$ $\quad ad(w)|_{\mathfrak{h}} : \mathfrak{h} \longrightarrow \mathfrak{h}$ $\quad$ pos ser $\mathfrak{h}$ ideal

f $\quad ad(w)|_{\mathfrak{h}}^{n}$ también, $\forall n \in \mathbb{N}$ luego

$\exp(ad(w)) : \mathfrak{h} \to \mathfrak{h}$

Ahora bien $\exp(\mathrm{ad}(w)) = \mathrm{Ad}_{\exp(w)}$. Veamos:

$\mathfrak{gl}(\mathfrak{g})$



$GL(\mathfrak{g})$

$\gamma : t \longmapsto \exp(t\, \mathrm{ad}(w))$

$\alpha : t \longmapsto \mathrm{Ad}_{\exp(tw)}$

① $\gamma(0) = \alpha(0) = id \in GL(\mathfrak{g})$

② $\gamma'(0) = \mathrm{ad}(w)$

$\alpha'(0) = \frac{d}{dt} \mathrm{Ad}_{\exp(tw)} : \mathfrak{g} \to \mathfrak{g} = \mathrm{ad}(w)$

$v \longmapsto \bar{v}$

$$\left\{ \bar{v} = \frac{d}{dt}\left(\mathrm{Ad}_{\exp(tw)}(v)\right) = \frac{d}{dt}\left(\frac{d}{ds} \exp(tw) \exp(sv) \exp(-tw)\right) = \mathrm{ad}(w)(v) \right\}$$

③ $\alpha(t_1) \circ \alpha(t_2)\, v = \mathrm{Ad}_{\exp(t_1 w)} \cdot \mathrm{Ad}_{\exp(t_2 w)}\, v = \mathrm{Ad}_{\exp(t_1 w)\cdot \exp(t_2 w)}(v) =$
$= \mathrm{Ad}_{\exp(t_1 + t_2) w}(v) = \alpha(t_1 + t_2)(v)$

Con ①, ② y ③ $\Rightarrow \gamma = \alpha \Rightarrow \exp(\mathrm{ad}(w)) = \mathrm{Ad}_{\exp(w)}$

$\mathfrak{h} \to \mathfrak{h}$

Seguimos:

$\mathrm{Ad}_{\exp(w)}(v) = \frac{d}{dt}\Big|_{t=0} \exp(w) \exp(tv) \exp(-w)$

$\exp(w) \exp(v) \exp(-w) = \exp(\underbrace{\mathrm{Ad}_{\exp(w)}(v)}_{\in \mathfrak{h} \text{ por lo de arriba}}) \in H$

$\boxed{\text{Idea: } \exp \text{ y } \frac{d}{dt}\Big|_{t=0} \text{ son "inversas"}}$

claim

Por lo tanto, $\exp(\mathfrak{h})$ es subgrupo normal de $\exp(\mathfrak{g})$.

Ahora por proposición 5.5.7 "cuadrado de valor" (sección Exponential map) tenemos el resultado (la aplic exponencial es sobreyectiva si el grupo es compacto y conexo).

Más compacto

Lemma 0.1. *Let G be a connected matrix Lie group, with (real) Lie algebra $\mathfrak{g}$, and $H < G$ a connected analytic subgroup with Lie algebra $\mathfrak{h} < \mathfrak{g}$. Then $H \lhd G \Leftrightarrow \mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

Proof. Suppose $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Then we may restrict $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$. So for $X \in \mathfrak{g}, Y \in \mathfrak{h}$, $ad_X^n(Y) \in \mathfrak{h}$. Thus, $e^{ad_X}(Y) \in \mathfrak{h}$. Then $e^X e^Y e^{-X} = exp(Ad_{e^X}(Y)) = exp(e^{ad_X}(Y)) \in exp(\mathfrak{h})$. So $exp(\mathfrak{g})$ normalizes $exp(\mathfrak{h})$. Since $G = \bigcup_{n\geq 0} exp(\mathfrak{g})^n$, $H = \bigcup_{n\geq 0} exp(\mathfrak{h})^n$, we see that G normalizes H.

Conversely, suppose that G normalizes H. Then G acts on H by conjugation. For $g \in G$, the derivative of this map is $Ad_g : T_eH \to T_eH = \mathfrak{h}$. Then for $X \in \mathfrak{g}$, $Ad_{e^{tX}} : \mathfrak{h} \to \mathfrak{h}$. Taking the derivative at $t = 0$, we see that $ad_X : \mathfrak{h} \to \mathfrak{h}$, so $\mathfrak{h}$ is an ideal (see Proposition 2.24). $\square$